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On the filling in holes problem for operator matrices ☆

Xiaoling Chen [a], Shifang Zhang [b,c,*], Huaijie Zhong [b]

[a] *School of Sciences, Jimei University, Xiamen 361021, PR China*

[b] *Department of Mathematics, Fujian Normal University, Fuzhou 350007, PR China*

[c] *Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China*

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We consider upper-triangular 2-by-2 operator matrices and are interested in the set that has to be added to certain spectra of the matrix in order to get the union of the corresponding spectra of the two diagonal operators. We show that in the cases of the Browder essential approximate point spectrum, the upper semi-Fredholm spectrum, or the lower semi-Fredholm spectrum the set in question need not to be an open set but may be just a singleton. In addition, we modify and extend known results on Hilbert space operators to operators on Banach spaces.

1. Introduction

If T is a Hilbert space operator and S is an invariant subspace for T, then T has the representation

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : S \oplus S^{\perp} \to S \oplus S^{\perp},$$

which motivates the interest in 2×2 upper-triangular operator matrices. See Refs. [1–15] for recent investigations on the subject.

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* Corresponding author. Address: Department of Mathematics, Zhejiang University, Hangzhou 310027, PR China.
E-mail address: shifangzhangfj@163.com (S. Zhang).

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For a given pair (A, B) of operators, Han et al. [9] considered the filling in holes problem. Their main result can be described as follows. If $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is an operator acting on a Banach space $X \oplus Y$, then

$$\sigma(A) \cup \sigma(B) = \sigma(M_C) \cup W,$$

where W is at the same time a subset of $\sigma(A) \cap \sigma(B)$ and a union of certain holes in $\sigma(M_C)$. For the Browder spectrum $\sigma_b(\cdot)$, the Weyl spectrum $\sigma_w(\cdot)$, the essential spectrum $\sigma_e(\cdot)$, and the Drazin spectrum $\sigma_D(\cdot)$ analogous results were obtained in [3,9,12,14,15]. In [10], Hwang and Lee also proved that if $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is an operator acting on a Hilbert space $H \oplus K$, then for the approximate point spectrum $\sigma_a(\cdot)$ we have

$$\sigma_a(A) \cup \sigma_a(B) = \sigma_a(M_C) \cup W,$$

where W is a subset of $\sigma_d(A) \cap \sigma_a(B)$ that is contained in the union of the holes in $\sigma_a(A)$.

The purpose of this note is to point out that part of the results of [1,2] are incorrect and to modify and extend some known results on the subject to the Banach space case.

Throughout this note, let X and Y be infinite dimensional Banach spaces, let $X \oplus Y$ be their direct sum, and let $B(X, Y)$ be the set of all bounded linear operators from X into Y. For simplicity, we also write $B(X, X)$ as $B(X)$.

For $T \in B(X, Y)$, we use $R(T)$ and $N(T)$ to denote the range and kernel of T, respectively. Let $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim Y/R(T)$. The ascent of $T \in B(X)$, denoted by $\mathrm{asc}(T)$, is defined as the smallest nonnegative integer k (if it exists) such that $N(T^k) = N(T^{k+1})$. If such k does not exist, we say that the ascent of T is equal to infinity. The descent of T, denoted by $\mathrm{des}(T)$, is defined as the smallest nonnegative integer k (if it exists) for which $R(T^k) = R(T^{k+1})$. If such k does not exist, we define the descent of T to be infinite. If the ascent and the descent of T are finite, then they are equal [5]. For $T \in B(X)$, if $R(T)$ is closed and $\alpha(T) < \infty$, we call T an upper semi-Fredholm operator and if $\beta(T) < \infty$, then T is called a lower semi-Fredholm operator. If T is a semi-Fredholm operator, then the index of T is defined by $i(T) = \alpha(T) - \beta(T)$.

The sets of all Fredholm operators, upper semi-Fredholm operators, lower semi-Fredholm operators, Weyl operators, upper semi-Weyl operators, lower semi-Weyl operators, Browder operators, upper semi-Browder operators and lower semi-Browder operators on X are defined by

$$\begin{aligned}
\Phi(X) &= \{T \in B(X) : \alpha(T) < \infty \text{ and } \beta(T) < \infty\},\\
\Phi_+(X) &= \{T \in B(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed}\},\\
\Phi_-(X) &= \{T \in B(X) : \beta(T) < \infty\},\\
\Phi_0(X) &= \{T \in \Phi(X) : i(T) = 0\},\\
\Phi_+^-(X) &= \{T \in \Phi_+(X) : i(T) \leqslant 0\},\\
\Phi_-^+(X) &= \{T \in \Phi_-(X) : i(T) \geqslant 0\},\\
\Phi_b(X) &= \{T \in \Phi(X) : \mathrm{asc}(T) = \mathrm{des}(T) < \infty\},\\
\Phi_{ab}(X) &= \{T \in \Phi_+(X) : \mathrm{asc}(T) < \infty\},\\
\Phi_{sb}(X) &= \{T \in \Phi_-(X) : \mathrm{des}(T) < \infty\}.
\end{aligned}$$

It is well-known that $\Phi_{ab}(X) \subseteq \Phi_+^-(X) \subseteq \Phi_+(X)$ and that $\Phi_{sb}(X) \subseteq \Phi_-^+(X) \subseteq \Phi_-(X)$. The corresponding spectra of an operator $T \in B(X)$ are defined as follows:

the essential spectrum: $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi(X)\}$;

the upper semi-Fredholm spectrum:

$$\sigma_{SF+}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(X)\};$$

the lower semi-Fredholm spectrum:

$$\sigma_{SF-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-(X)\};$$

the Weyl spectrum: $\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_0(X)\}$;
the upper semi-Weyl spectrum: $\sigma_{aw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+^-(X)\}$;
the lower semi-Weyl spectrum: $\sigma_{sw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_-^+(X)\}$;
the Browder spectrum: $\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_b(X)\}$;
the Browder essential approximate point spectrum:

$$\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{ab}(X)\};$$

the lower semi-Browder spectrum: $\sigma_{sb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_{sb}(X)\}$; the Kato spectrum: $\sigma_K(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_+(X) \cup \Phi_-(X)\}$.

Moreover, the spectrum, the approximate point spectrum, and the defect spectrum of $T \in B(X)$ are given by

$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible$\}$;
$\sigma_a(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ is not bounded below$\}$;
$\sigma_d(T) = \{\lambda \in \mathbb{C} : R(T - \lambda I) \neq X\}$.

An operator $T \in B(X)$ is called Drazin invertible if there exists an operator $T^D \in B(X)$ such that $TT^D = T^DT, T^DTT^D = T^D, T^{k+1}T^D = T^k$ for some nonnegative integer k. In that case T^D is called a Drazin inverse of T [14]. The Drazin spectrum $\sigma_D(T)$ of T is defined as the set of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ is not Drazin invertible. Finally, for $A \in B(X)$, $B \in B(Y)$, $C \in B(Y, X)$, we put

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in B(X \oplus Y).$$

2. A counterexample

Let H be the direct sum of countably many copies of $\ell^2 := \ell^2(\mathbb{N})$. Thus, the elements of H are the sequences $\{x_j\}_{j=1}^{\infty}$ with $x_j \in \ell^2$ and $\sum_{j=1}^{\infty} \|x_j\|^2 < \infty$. Put $K = \ell^2$. Let V be the forward shift on ℓ^2,

$$V : \ell^2 \to \ell^2, \quad \{z_1, z_2, \ldots\} \mapsto \{0, z_1, z_2, \ldots\},$$

define the operators A and C by

$$A : H \to H, \quad \{x_1, x_2, \ldots\} \mapsto \{Vx_1, Vx_2, \ldots\}$$

and

$$C : K \to H, \quad \{y_1, y_2, \ldots\} \mapsto \{y_1 e_1, y_2 e_1, \ldots\}$$

where $e_1 = \{1, 0, 0, \ldots\}$, and consider the operator

$$M_C = \begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix} : H \oplus K \to H \oplus K.$$

Note that

$$M_C : (\{x_1, x_2, \ldots\}, y) \mapsto (\{y_1 e_1 + Vx_1, y_2 e_1 + Vx_2, \ldots\}, 0).$$

Let $\mathbb{D}$, $\overline{\mathbb{D}}$, and $\mathbb{T}$ denote open unit disk, the closed unit disk, and the unit circle, respectively. For a compact subset M of $\mathbb{C}$, we use ∂M to denote the boundary of M.

(i) Taking into account that $V - \lambda I$ is an injective Fredholm operator with index -1 for every $\lambda \in \mathbb{D}$, we see that $A - \lambda I$ is an injective upper semi-Fredholm operator with index of $-\infty$ for every $\lambda \in \mathbb{D}$. It follows that $M_C - \lambda I$ is an injective upper semi-Fredholm operator with index $-\infty$ for every $\lambda \in \mathbb{D}$ and that $\sigma(M_C) = \sigma(A) = \overline{\mathbb{D}}$.

(ii) A straightforward calculation shows that

$$\sigma_{SF+}(A) = \sigma_{aw}(A) = \sigma_{ab}(A) = \sigma_a(A) = \partial(\sigma(A)) = \mathbb{T}$$

and that

$$\sigma_{SF+}(M_C) = \sigma_{aw}(M_C) = \sigma_{ab}(M_C) = \sigma_a(M_C) = \partial(\sigma(M_C)) = \mathbb{T}.$$

Consequently, if $\sigma_\tau \in \{\sigma_a, \sigma_{SF+}, \sigma_{aw}, \sigma_{ab}\}$, then

$$\sigma_\tau(A) \cup \sigma_\tau(0) = \sigma_\tau(M_C) \cup \{0\} = \mathbb{T} \cup \{0\}, \sigma_\tau(M_C) \cap \sigma_\tau(0) = \mathbb{T} \cap \{0\} = \emptyset.$$

(iii) Although $\sigma_d(A) \cap \sigma_{ab}(0)$ has no interior points (in fact, $\sigma_d(A) \cap \sigma_{ab}(0) = \{0\}$), we have

$$\begin{aligned}(\sigma_a(A) \cup \sigma_a(0)) \setminus \sigma_a(M_C) &= (\sigma_{SF+}(A) \cup \sigma_{SF+}(0)) \setminus \sigma_{SF+}(M_C) \\ &= (\sigma_{ab}(A) \cup \sigma_{ab}(0)) \setminus \sigma_{ab}(M_C) = \{0\}.\end{aligned}$$

The set $\sigma_{SF-}(A) \cap \sigma_{SF+}(0)$ has no interior points (in fact, $\sigma_{SF-}(A) \cap \sigma_{SF+}(0) = \{0\}$), but

$$(\sigma_{SF+}(A) \cup \sigma_{SF+}(0)) \setminus \sigma_{SF+}(M_C) = \{0\}.$$

This example illustrates that if $\sigma_* \in \{\sigma_a, \sigma_{SF+}, \sigma_{aw}, \sigma_{ab}\}$, then it is just a point and thus not an open set from $\sigma_*(A) \cup \sigma_*(B)$ to $\sigma_*(M_C)$:

$$\sigma_*(A) \cup \sigma_*(B) = \sigma_*(M_C) \cup \{0\}.$$

Thus, neither Theorem 2.3 nor Corollary 2.4 in [1] is always true and Theorems 3.1 and 3.2 and Corollary 3.4 in [2] do not always hold, since $\sigma_{SF+}(\cdot)$ and $\sigma_{SF-}(\cdot)$ are dual. Moreover, the example also tells us that the claim $\partial\sigma_{ab}(B) \subseteq \sigma_{ab}(M_C)$, which appears in line 21 on page 66 in [1], is not always true.

Here is another remark. Let $\eta(\cdot)$ denote the polynomially convex hull. In [9], for a given pair (A, B) of operators, after proving that

$$\partial(\sigma(A) \cup \sigma(B)) \subseteq \sigma(M_C) \subseteq \sigma(A) \cup \sigma(B),$$

which implies that $\eta(\sigma(M_C)) = \eta(\sigma(A) \cup \sigma(B))$, the authors show that $\sigma(A) \cup \sigma(B) = \sigma(M_C) \cup W$ where W is a union of certain holes in $\sigma(M_C)$. We want to draw the reader's attention to the fact for other spectra $\sigma_*(\cdot)$, even if both

$$\sigma_*(M_C) \subseteq \sigma_*(A) \cup \sigma_*(B) \tag{1}$$

and

$$\eta(\sigma_*(M_C)) = \eta(\sigma_*(A) \cup \sigma_*(B)) \tag{2}$$

hold, which implies that $\sigma_*(A) \cup \sigma_*(B) = \sigma_*(M_C) \cup W$, where W is contained in the union of the holes in $\sigma_*(M_C)$, it need not to be true

$$\sigma_*(A) \cup \sigma_*(B) = \sigma_*(M_C) \cup W,$$

where W is the union of certain holes in $\sigma_*(M_C)$. Indeed, the example given above shows that W may be a singleton and hence reveals that (1) and (2) do not even suffice to guarantee that W is an open set.

3. Modifications and extensions

The passage from $\sigma_*(A) \cup \sigma_*(B)$ to $\sigma_*(M_C)$ was studied in [1,2] on Hilbert spaces in the case where $\sigma_* \in \{\sigma_{SF+}, \sigma_{SF-}, \sigma_{ab}\}$. We will modify and extend these results to Banach spaces. Throughout what follows X, Y are Banach spaces and $A \in B(X)$, $B \in B(Y)$, $C \in B(Y, X)$.

Lemma 3.1 [9,12,14,15]. *If* $\sigma_* \in \{\sigma, \sigma_b, \sigma_w, \sigma_e, \sigma_D\}$, *then*

$$\eta(\sigma_*(A) \cup \sigma_*(B)) = \eta(\sigma_*(M_C))$$

and

$$\sigma_*(A) \cup \sigma_*(B) = \sigma_*(M_C) \cup W_*,$$

where W_* *is simultaneously a subset of* $\sigma_*(A) \cap \sigma_*(B)$ *and the union of certain holes in* $\sigma_*(M_C)$.

For $\sigma_w(\cdot)$, although the above result was only proved for Hilbert spaces, it is also true on Banach spaces.

Lemma 3.2. *Let* $\sigma_* \in \{\sigma_w, \sigma_{ab}, \sigma_e, \sigma_{SF+}, \sigma_{SF-}\}$. *Then*

$$\eta(\sigma_b(A) \cup \sigma_b(B)) = \eta(\sigma_b(M_C)) = \eta(\sigma_*(A) \cup \sigma_*(B)) = \eta(\sigma_*(M_C))$$

and

$$\sigma_*(M_C) \subseteq \sigma_*(A) \cup \sigma_*(B). \tag{3}$$

Proof. First of all, one has $\eta(\sigma_b(T)) = \eta(\sigma_*(T))$ for every $T \in B(X)$ if $\sigma_* \in \{\sigma_w, \sigma_{ab}, \sigma_e, \sigma_{SF+}, \sigma_{SF-}, \sigma_K\}$. Indeed, it is well known that

$$\partial(\sigma_b(T)) \subseteq \sigma_K(T) \subseteq \sigma_*(T) \subseteq \sigma_b(T)$$

for $\sigma_* \in \{\sigma_w, \sigma_{ab}, \sigma_e, \sigma_{SF+}, \sigma_{SF-}, \sigma_K\}$. This implies that $\eta(\sigma_b(T)) = \eta(\sigma_*(T))$. Similarly, we have

$$\eta(\sigma_b(A) \cup \sigma_b(B)) = \eta(\sigma_*(A) \cup \sigma_*(B))$$

for every pair (A, B) of operators if $\sigma_* \in \{\sigma_w, \sigma_{ab}, \sigma_e, \sigma_{SF+}, \sigma_{SF-}, \sigma_K\}$. Using Lemma 3.1, we get

$$\eta(\sigma_*(A) \cup \sigma_*(B)) = \eta(\sigma_b(A) \cup \sigma_b(B)) = \eta(\sigma_b(M_C)) = \eta(\sigma_*(M_C)),$$

for $\sigma_* \in \{\sigma_w, \sigma_{ab}, \sigma_e, \sigma_{SF+}, \sigma_{SF-}, \sigma_K\}$.

Using the identity

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix},$$

it is not hard to show inclusion (3). □

We remark that inclusion (3) does in general not hold for $\sigma_K(\cdot)$. To see this, let $A, B \in B(\ell^2)$ be defined by

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots),$$
$$B(x_1, x_2, x_3, \ldots) = (x_1, x_3, x_5, \ldots)$$

and let $C = 0$. Then

$$0 \in \sigma_K(M_C), \quad 0 \notin \sigma_K(A) \cup \sigma_K(B).$$

Accordingly, $\sigma_K(M_C) \nsubseteq \sigma_K(A) \cup \sigma_K(B)$.

Theorem 3.3. *We have*

$$\sigma_{SF+}(A) \cup \sigma_{SF+}(B) = \sigma_{SF+}(M_C) \cup W_{SF+}$$

where $W_{SF+} \subseteq [(\sigma_{SF+}(B) \cap \sigma_{SF-}(A)) \setminus \sigma_{SF+}(A)]$ *is not only contained in certain holes in* $\sigma_{SF+}(A)$ *but also in certain holes in* $\sigma_{SF+}(M_C)$.

Proof. Similar to the proof of Theorem 3.1 in [2], we have that

$$\sigma_{SF+}(A) \cup \sigma_{SF+}(B) = \sigma_{SF+}(M_C) \cup W_{SF+},$$

where W_{SF+} is contained in certain holes in $\sigma_{SF+}(M_C)$ which in turn are subsets of $\sigma_{SF-}(A) \cap \sigma_{SF+}(B)$. Thus, in order to prove that $W_{SF+} \subseteq [(\sigma_{SF+}(B) \cap \sigma_{SF-}(A)) \setminus \sigma_{SF+}(A)]$, we only need to prove that $\sigma_{SF+}(A) \subseteq \sigma_{SF+}(M_C)$. For this it is sufficient to show that A is a lower semi-Fredholm operator if M_C is upper semi-Fredholm, which is obvious. Therefore $W_{SF+} \subseteq [(\sigma_{SF+}(B) \cap \sigma_{SF-}(A)) \setminus \sigma_{SF+}(A)]$. Next, we can claim that W_{SF+} is contained in the union of the holes in $\sigma_{SF+}(M_C)$. In fact, from Lemma 3.2 we deduce that

$$\eta(\sigma_{SF+}(M_C)) = \eta(\sigma_b(M_C)) = \eta(\sigma_b(A) \cup \sigma_b(B)) = \eta(\sigma_{SF+}(A) \cup \sigma_{SF+}(B)).$$

Moreover, since $W_{SF+} \subseteq \sigma_{SF+}(A) \cup \sigma_{SF+}(B)$ by what was proved above, it follows that $W_{SF+} \subseteq \sigma_{SF+}(A) \cup \sigma_{SF+}(B) \subseteq \eta(\sigma_{SF+}(M_C))$. Accordingly, W_{SF+} is a subset of the union of the holes in $\sigma_{SF+}(M_C)$. □

As σ_{SF+} and σ_{SF-} are dual, we obtain the following result from the preceding theorem.

Theorem 3.4. *We have*

$$\sigma_{SF-}(A) \cup \sigma_{SF-}(B) = \sigma_{SF-}(M_C) \cup W_{SF-},$$

nowhere $W_{SF-} \subseteq [(\sigma_{SF-}(A) \cap \sigma_{SF+}(B)) \setminus \sigma_{SF-}(B)]$ *is not only a subset of the union of the holes in* $\sigma_{SF-}(B)$ *but also a subset of the union of the holes in* $\sigma_{SF-}(M_C)$.

Note that $\mathrm{asc}(M_C) < \infty$ implies that $\mathrm{asc}(A) < \infty$ and that $M_C \in \Phi_+(X \oplus Y)$ yields that $A \in \Phi_+(X)$. Combining these two facts with Lemma 3.2, it not hard to prove the following theorem.

Theorem 3.5. *We have*

$$\sigma_{ab}(A) \cup \sigma_{ab}(B) = \sigma_{ab}(M_C) \cup W_{ab},$$

where $W_{ab} \subseteq (\sigma_{ab}(B) \setminus \sigma_{ab}(A))$ *is contained in the union of the holes in* $\sigma_{ab}(M_C)$.

By duality, we arrive at the following.

Theorem 2.7. *We have*

$$\sigma_{sb}(A) \cup \sigma_{sb}(B) = \sigma_{sb}(M_C) \cup W_{sb},$$

where $W_{sb} \subseteq (\sigma_{sb}(A) \setminus \sigma_{sb}(B))$ *is a subset of the union of the holes in* $\sigma_{sb}(M_C)$.

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